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ASYMPTOTIC METHODS OF SOLVING TWO-DIMENSIONAL DYNAMIC PROBLEMS OF A VISCOELASTIC LAYER WITH MIXED BOUNDARY CONDITIONS*

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Problems of shear of a viscoelastic layer by a rigid punch and of pressing the latter into such layer lying on a viscoelastic Winkler foundation are considered. The punch is subjected to time dependent harmonic forces. The deformation model is defined by the three-constants law (conventional body). Similar problems are considered in the case of an elastic layer on an elastic Winkler foundation. All these problems are first reduced to integral equations of the first kind and, then, to infinite algebraic systems in conformity with /1,2/ that are solvable for small values of the characteristic geometric parameter. An asymptotic method for large values of that parameter is also developed. Similar methods were considered in /3, 4/ in the case of a large characteristic parameter.

1. Let us consider the problem of a viscoelastic layer of thickness h lying on a viscoelastic Winkler foundation and subjected to shear vibration by a rigid strip punch of width $2a$. A shear force $T = T_0 \exp(-i\omega t)$ is applied to the punch. Equations of the linear theory of viscoelasticity expressed in terms of displacements are of the form /5/

$$\int_{-\infty}^{t} \mu(t-\tau)\frac{\partial u_{i,ll}}{\partial \tau}\,d\tau + \int_{-\infty}^{t} [\lambda(t-\tau)+\mu(t-\tau)]\frac{\partial u_{k,ki}}{\partial \tau}\,d\tau = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad i=1,2,3 \tag{1.1}$$

$$\sigma_{ij} = \delta_{ij}\int_{-\infty}^{t} \lambda(t-\tau)\frac{\partial \varepsilon_{kk}(\tau)}{\partial \tau}\,d\tau + 2\int_{-\infty}^{t} \mu(t-\tau)\frac{\partial \varepsilon_{ij}(\tau)}{\partial \tau}\,d\tau$$

where $\lambda(t)$ and $\mu(t)$ are relaxation functions and ρ is the volume density of the layer material.

Boundary conditions of the problem are

$$y=0:\ -\rho_*\frac{\partial^2 u_3}{\partial t^2} + \sigma_{23} = \int_{-\infty}^{t} \nu(t-\tau)\frac{\partial u_3}{\partial \tau}\,d\tau, \quad |x|<\infty \tag{1.2}$$

$$y=h:\ \sigma_{23}=0, \quad |x|>a; \quad u_3 = \bar\varepsilon(x)\exp(-i\omega t), \quad |x|\leqslant a \tag{1.3}$$

$$u_1 = u_2 \equiv 0; \quad \sigma_{13}\to 0, \ |x|\to\infty$$

of which (1.2) defines the work of the viscoelastic Winker foundation, where ρ^* is the surface density of the base material and $\nu(t)$ is the relaxation function. Below, when deriving specific formulas we use, without loss of generality, the three constants law of linear deformation, viz. assume the layer shear modulus to be of the form /5/

$$\mu(t) = G_0 + G_1\exp(-t/t_1) \tag{1.4}$$

where G_0 and G_1 are the static and instantaneous shear moduli, respectively, and t_1 is the relaxation time. In condition (1.2) we similarly assume

$$\nu(t) = k_0 + k_1\exp(-t/t_2) \tag{1.5}$$

where k_0 and k_1 are the static and instantaneous coefficients of the foundation and t_2 is the relaxation time.

We seek a solution of the form $u_3(x, y, t) = \bar u_3(x, y)\exp(-i\omega t)$ of the problem, and applying to Eqs.(1.1) the integral Fourier transform with respect to x reduce the boundary value problem to the integral equation of the first kind

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$$\int_{-a}^{a} \tau(\xi) k(x-\xi) d\xi = \varepsilon(x), \quad |x| \leqslant a \tag{1.6}$$

$$k(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(u) \exp(iut) du$$

$$\tau(\xi) = \bar{\tau}(\xi h) \bar{\mu}^{-1}, \quad \varepsilon(x) = \bar{\varepsilon}(xh) h^{-1}, \quad a' = \lambda^{-1} = ah^{-1}$$

where $\bar{\tau}(x)$ is the amplitude of the unknown contact shear stresses under the punch, $\bar{\mu}$ is the complex shear modulus /5/ in which the prime at the dimensionless quantity a' is omitted here and subsequently.

The Fourier transform of kernel $K(u)$ is of the form

$$K(u) = \frac{1 + \lambda_0 \sigma^{-1} \operatorname{th} \sigma}{\lambda_0 + \sigma \operatorname{th} \sigma}, \quad \lambda_0 = \frac{(\bar{\nu} - \rho_* \omega^2) h}{\bar{\mu}} \tag{1.7}$$

$$\sigma = \sqrt{u^2 - b^2}, \quad b^2 = \rho \omega^2 h^2 \bar{\mu}^{-1}$$

$$\bar{\mu} = G_0 - G_1 \frac{i\omega t_1}{1 - i\omega t_1}, \quad \bar{\nu} = k_0 - k_1 \frac{i\omega t_2}{1 - i\omega t_2}$$

The plane problem of impression of a vibrating rigid punch of width $2a$ into a viscoelastic layer of thickness h lying without friction on a viscoelastic foundation of the Winkler type is similarly formulated. Boundary conditions of such problem are of the form

$$y = 0: \quad -\rho_* \frac{\partial^2 u_2}{\partial t^2} + \sigma_{22} = \int_{-\infty}^{t} \nu(t-\tau) \frac{\partial u_2}{\partial \tau} d\tau, \quad |x| < \infty \tag{1.8}$$

$$\sigma_{12} = 0, \quad |x| < \infty$$

$$y = h: \quad \sigma_{12} = 0, \quad |x| < \infty$$

$$\sigma_{22} = 0, \quad |x| > a; \quad u_2 = \bar{\varepsilon}(x) \exp(-i\omega t), \quad |x| \leqslant a$$

$$u_3 \equiv 0; \quad \sigma_{11}, \sigma_{12} \to 0, \quad |x| \to \infty$$

Here and in (1.1)

$$\lambda(t) = G_\lambda^0 + G_\lambda^1 \exp(-t/t_0), \quad \mu(t) = G_\mu^0 + G_\mu^1 \exp(-t/t_1) \tag{1.9}$$

$$\nu(t) = k_0 + k_1 \exp(-t/t_2)$$

$\mu(t)$, $\lambda(t)$, $\nu(t)$ is the relaxation function, G_λ^0, G_λ^1, t_0, G_μ^0, G_μ^1, t_1 and k_0, k_1, t_2 are, respectively, the static and instantaneous moduli and the relaxation time of functions λ, μ and ν.

Conditions (1.8) with Eqs.(1.1) define the mixed boundary value problem. Using the representation $u_i(x, y, t) = \bar{u}_i(x, y) \exp(-i\omega t)$ and applying the integral Fourier transform in x, we obtain for the solution of the boundary value problem the integral equation of the problem defined by (1.6) in which it is necessary to substitute the dimensionless amplitude $q(\xi)$ of contact pressure for $\tau(\xi)$. Function $K(u)$ is not presented here owing to its unwieldiness.

In the theoretical plane the problems of shear vibration and of vibrating impression of a rigid punch into an elastic layer on an elastic Winkler foundation are interesting in themselves. The first of these is reduced in conformity with the scheme described above to the integral equation (1.6) in which

$$K(u) = (1 + \lambda_0 \sigma^{-1} \operatorname{th} \sigma)(\lambda_0 + \sigma \operatorname{th} \sigma)^{-1} \tag{1.10}$$

$$\sigma = \sqrt{u^2 - \varkappa^2}, \quad \varkappa^2 = \rho \omega^2 h^2 \mu^{-1}, \quad \lambda_0 = (k - \rho_* \omega^2) h \mu^{-1}$$

where k is the coefficient of the Winkler foundation and μ is the shear modulus of the layer material. For brevity, the expression for $K(u)$ is not written out here.

2. Let us consider the integral equation (1.6) on the assumption that $K(u)$ can be represented in the form

$$K(u) = K(0) \prod_{n=1}^{\infty} \left(1 + \frac{u^2}{\delta_n^2}\right)\left(1 + \frac{u^2}{\gamma_n^2}\right)^{-1} = K(0) P_1(u^2) P_2^{-1}(u^2) \tag{2.1}$$

where $\pm i\delta_n$, $\pm i\gamma_n$ is a denumerable set of simple zeros and poles in the complex plane $(u = \sigma + i\tau)$, a finite number of which may lie on the real axis. In this case integration along the real axis in the formula for $k(t)$ in (1.6) must be replaced by integration along countour Γ which with the limit absorption principle taken into account in the conventional (regular) case deviates from the real axis only where it bypasses the negative and positive poles from above and below, respectively, /6/. Let, moreover, δ_n and γ_n increase monotonically with increasing subscript number thus ensuring the convergence of the infinite product (2.1), and let along any regular system of contours C_n the estimate

$$K(u) = O(|u|^{-1}), \quad u \to \infty \tag{2.2}$$

holds in the complex plane.

Let us consider the case of $\varepsilon(x) = \exp(-\varepsilon x)$ on the assumption that function $\varepsilon(x)$ may be represented by a Fourier integral. In conformity with /1,2/ the solution of the integral equation (1.6) can be of the form

$$\tau(x) = K^{-1}(i\varepsilon)\exp(-\varepsilon x) + \sum_{n=1}^{\infty} H_n(x), \quad H_n(x) = C_n\exp(-\delta_n x) + D_n\exp(\delta_n x) \tag{2.3}$$

Substituting (2.3) into (1.6) and taking the necessary quadratures with allowance for (2.1), (2.3), and Jordan's lemma, we obtain

$$\sum_{m=1}^{\infty} \frac{P_1(-\gamma_m^2)}{P_2'(-\gamma_m^2)}[\zeta(\gamma_m)\exp(-\gamma_m x) + \eta(\gamma_m)\exp(\gamma_m x)]\exp(-\gamma_m a) = 0 \tag{2.4}$$

Taking into consideration the linear independence of functions $\exp(-\gamma_m x)$ and $\exp(\gamma_m x)$ /7/, for the determination of C_n and D_n we obtain from (2.4) the infinite algebraic system

$$\frac{\exp(\varepsilon a)}{K(i\varepsilon)(\varepsilon - \gamma_m)} + \sum_{n=1}^{\infty}\left[C_n\frac{\exp(\delta_n a)}{\delta_n - \gamma_m} - D_n\frac{\exp(-\delta_n a)}{\delta_n + \gamma_m}\right] = 0, \quad m = 1, 2, \ldots \tag{2.5}$$

$$\frac{\exp(-\varepsilon a)}{K(i\varepsilon)(\varepsilon + \gamma_m)} + \sum_{n=1}^{\infty}\left[C_n\frac{\exp(-\delta_n a)}{\delta_n + \gamma_m} - D_n\frac{\exp(\delta_n a)}{\delta_n - \gamma_m}\right] = 0, m = 1, 2\ldots$$

which in the case of a flat punch $(\varepsilon = 0)$ reduces to

$$\sum_{n=1}^{\infty} a_{mn}x_n = -\gamma_m^{-1}, \quad m = 1, 2, \ldots \tag{2.6}$$

$$a_{mn} = (\gamma_m + \delta_n \operatorname{th}\delta_n a)(\gamma_m^2 + \delta_n^2)^{-1}$$

and solution (2.3) assumes the form

$$\tau(x) = K^{-1}(0)\left(1 + \sum_{n=1}^{\infty} x_n \operatorname{ch}\delta_n x \operatorname{ch}^{-1}\delta_n a\right), \quad |x| \leqslant a \tag{2.7}$$

This solution is valid for a small parameter $\lambda = h/a$. The infinite algebraic systems (2.5) and (2.6) were investigated in /6,8/.

3. We shall now derive a solution of the integral equation (1.6) which is effective for high values of parameter λ. We assume that $K(u)$ in (1.6) has N poles on the real axis and that integration is carried out along contour Γ defined in Sect.2. Taking into account the selected integration contour, we represent function $k(t)$ as

$$k(t) = \sum_{m=1}^{N} \frac{P_1(-\gamma_m^2)}{P_2'(-\gamma_m^2)}\cos(i\gamma_m t) + \text{v. p. } \frac{1}{2\pi}\int_{-\infty}^{\infty} K(u)e^{iut}du \tag{3.1}$$

where γ_m $(m = 1, 2, \ldots, N)$ are poles of $K(u)$ along the positive side of the real axis.

Let $K(u) = u^{-1}L(u)$. Then, separating in the integral in (3.1) the singularity at $t \to 0$ and regularizing it on the real axis, taking into account the evenness of function $K(u)$ and the following asymptotic estimates for $L(u)$:

$$L(u) = 1 + c_1u^{-2} + c_2u^{-4} + O(u^{-6}), \quad u \to \infty \tag{3.2}$$

$$L(u) = K(0)u + O(u^3), \quad u \to 0$$

we obtain

$$\pi k(t) = -\ln|t| + \pi \sum_{m=1}^{N} \frac{P_1(-\gamma_m^2)}{P_2'(-\gamma_m^2)} e^{-\gamma_m|t|} + a_{30}^* + a_{20}^*|t| + a_{11}^* t^2 \ln|t| + a_{31}^* t^2 + a_{21}^*|t|^3 + O(t^4) = \tag{3.3}$$

$$-\ln|t| + a_{30} + a_{20}|t| + a_{11}t^2 \ln|t| + a_{31}t^2 + a_{21}|t|^3 + O(t^4)$$

$$a_{30} = \pi i \sum_{m=1}^{N} \frac{P_1(-\gamma_m^2)}{P_2'(-\gamma_m^2)} + \int_0^\infty u^{-1}\left[L(u) - 1 - \sum_{m=1}^{N} \frac{A_m u}{u^2+\gamma_m^2} + e^{-u}\right] du$$

$$a_{20} = -2\pi i \sum_{m=1}^{N} \frac{P_1(-\gamma_m^2)}{P_2'(-\gamma_m^2)} \gamma_m, \quad a_{31} = -\frac{3}{4}c_2 - \frac{1}{2}\int_0^\infty F(u)\,du, \quad a_{11} = \frac{c_2}{2}, \quad a_{21} = 0$$

$$F(u) = u^{-1}\left[L(u) - 1 - \sum_{m=1}^{N} \frac{A_m u}{u^2+\gamma_m^2} + u^{-1}\sum_{m=1}^{N} A_m - c_2(1-e^{-u})\right]$$

$$A_m = 2\gamma_m i \frac{P_1(-\gamma_m^2)}{P_2'(-\gamma_m^2)}$$

Having obtained the asymptotic expansion (3.3) of kernel $k(t)$ for small t, we construct the solution of Eq.(1.6) using the method of /9/. We have

$$\tau(x) = (a^2 - x^2)^{-1/2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \omega_{ij}(x)\lambda^{-m}\ln^n\lambda \tag{3.4}$$

where function $\omega_{ij}(x)$ is obtained from formulas (1.13) and, in the particular case of $\varepsilon(x) = \varepsilon$ from formulas (1.14)—(1.18) of /9/. Note that the method of effective when $\lambda > \max(\gamma_m)$, $m = 1,2,\ldots,N$.

4. Let us derive the formulas required for calculating the complex amplitude of displacement waves away from the punch. It is defined by the contour integral

$$h^{-1}\bar{u}_3(x) = \frac{1}{2\pi}\int_\Gamma \tau^*(u) K(u) e^{iux}\,du \tag{4.1}$$

where $\tau^*(u)$ is the Fourier transform of function $\tau(\xi)$.

Contour Γ was defined above. Closing the integration contour in the upper half-plane with allowance for (2.1)—(2.3), after some calculations, we obtain

$$h^{-1}\bar{u}_3(x) = 2i\sum_{m=1}^{\infty} D(\gamma_m)\left\{\frac{H(\varepsilon, -\gamma_m)}{K(i\varepsilon)} + \sum_{n=1}^{\infty}[C_n H(\delta_n, -\gamma_m) + D_n H(\delta_n, \gamma_m)]\right\} e^{-\gamma_m x} \tag{4.2}$$

$$H(u, v) = (u+v)^{-1}\,\mathrm{sh}\,(u+v)\,a, \quad D(\gamma_m) = P_1(-\gamma_m^2)\times[P_2'(-\gamma_m^2)]^{-1}$$

which in the case of a plane punch $(\varepsilon(x) = \varepsilon)$ becomes

$$h^{-1}\bar{u}_3(x) = \frac{2i}{K(0)}\sum_{m=1}^{\infty} D(\gamma_m)\left[\frac{\mathrm{sh}\,\gamma_m a}{\gamma_m} + \mathrm{ch}\,\gamma_m a\sum_{m=1}^{\infty} b_{mn}x_n\right] e^{-\gamma_m x}, \quad x > a \tag{4.3}$$

$$b_{mn} = (\gamma_m^2 - \delta_n^2)^{-1}[\gamma_m\,\mathrm{th}\,\gamma_m a - \delta_n\,\mathrm{th}\,\delta_n a]$$

where x_n is the same as in (2.6) and (2.7). These formulas are valid in the case of small λ. For large λ it is necessary to use solutions of the integral equation (1.6) of form (3.4). In that case $\tau^*(u)$ is determined by formula

$$\tau^*(u) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \omega_{mn}^*(u)\lambda^{-m}\ln^n\lambda \tag{4.4}$$

$$\omega_{mn}^*(u) = \frac{1}{\pi}\int_{-a}^{a}(a^2 - x^2)^{-1/2}\omega_{mn}(x)e^{-iux}\,dx$$

The general solution is very cumbersome, but for $\varepsilon(x) = \varepsilon$ it can be represented in the form

$$\omega_{00}^*(u) = PJ_0, \quad \omega_{10}^*(u) = 4\pi^{-3}a_{20}PS_1(u) \tag{4.5}$$

$$\omega_{20}^*(u) = \frac{P}{\pi}\left\{-\pi\left[a_{11}\left(\frac{3}{2} - \ln 2\right) + a_{31}\right] l(u) + 32\pi^4 a_{20}^2[S_2(u) - \pi 0.1508 J_0]\right\}, \quad \omega_{21}^*(u) = Pa_{11}l(u)$$

$$S_1(u) = \pi\left[l(u) + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2}(J_{2k+2} + J_{2k})\right]$$

$$S_2(u) = \pi\{0.4356(au)^{-1}J_1 - 0.1321(au)^{-3}[3aul(u) - (au)^2 J_1] - 0.4988\pi^{-1}\theta(u)\}$$

$$\theta(u) = \sum_{k=1}^{\infty} B(^3/_2, k + {}^1/_2) M(k-2, u)(2k-1)^{-1}$$

$$l(u) = -J_0 + 2(au)^{-1}J_1,\quad M(k, u) = F(k + {}^5/_2;\ {}^5/_2;\ k+4; -(au)^2/16)$$

where $J_m = J_m(au)$ $(m = 0, 1, 2, \ldots)$ is the Bessel function and P is obtained using formula (1.18) of /10/.

Substituting the expression for $\tau^*(u)$ in (4.4) into (4.1) and closing the integration contour in the upper half-plane with allowance for (2.1) and (2.2) we obtain

$$h^{-1}\bar{u}_3(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda^{-m}\ln^n\lambda\sum_{k=1}^{\infty}\frac{P_1(-\gamma_k^2)}{P_2'(-\gamma_k^2)}\omega_{mn}^*(i\gamma_k)\exp(-\gamma_k x),\quad x > a \tag{4.6}$$

Similar formulas can also be obtained for amplitudes at $x < -a$.

5. It can be shown that in the problem of viscoelastic (elastic) layer lying on a viscoelastic (elastic) Winkler foundation and subjected to shear vibration by a rigid punch all of properties (2.1), (2.2), and (3.2) are satisfied by function $K(u)$. Hence it is possible to use formula (2.7) for constructing a solution for small λ. To do this it is necessary to know the zeros and poles of $K(u)$ in the complex plane $(u = \sigma + i\tau)$. To obtain a qualitative picture of the phenomenon it is necessary to investigate the dependence of the amplitude of contact shear stresses $\tau(x)$ on parameters of the viscoelastic problem. When $|x| \ll 1$ and $h/a \ll 1$, it is possible to substitute for (2.7) the formula

$$\tau(x) = K^{-1}(0) + O(\exp(-(1-x)\delta_n a)) \tag{5.1}$$

For simplicity we set below $\rho_* = \rho h$, $t_1 = t_2$ and vary within wide limits the following parameters of the problem:

$$\varkappa^2 = \rho h^2\omega^2 G_0^{-1},\quad \delta^2 = G_0 t_1^2(\rho h^2)^{-1},\quad \theta = G_0/hk \tag{5.2}$$

$$\eta_1 = k_0(k_0 + k_1)^{-1},\quad \eta_2 = G_1 G_0^{-1}$$

Curves of $|\tau(x)|$ calculated by formula (5.1) in terms of the dimensionless frequency $\varkappa$ for x approaching zero when $\eta_1 = \eta_2 = \theta = 1$ are shown in Fig.1 for two values of δ indicated at respective curves. A simultaneous and equal relative change of the layer and Winkler foundation rigidity results in a slight shift of the resonant frequency to the left and a decrease of the amplitude of $|\tau(x)|$. When $\delta > 100$ the amplitude and resonance frequencies become stabilized (elasticity). The appearance of intermediate peaks $|\tau|$; when $\delta = 5$ is interesting; when $\delta \geqslant 10$ these peaks are absent.

Variation of parameter θ shows that when the layer is relatively rigid ($\delta = 100$), variation of the foundation rigidity only slightly affects the pattern of resonance frequency distribution and the amplitude. With a less rigid layer ($\delta = 5$) an increase of the foundation rigidity relative to that of the layer shifts the first resonance frequencies to the right, while for $\varkappa > 15$ the resonance frequencies are the same for various values of θ.

Let us investigate the effect of internal friction on the system operation mode. In the case of a layer more rigid than the foundation ($\theta = 100$) the variation of friction in the Winkler foundation (variation of η_1) is apparent only at low frequencies, and when $\varkappa > 0.2$ the amplitudes are independent of η_1. The layer internal friction η_2 has a considerable effect on the system resonant frequencies. Curves *1, 2*, and *3* in Fig.2 correspond to η_2 equal 0.1, 1, and 10 with $\delta = 5, \theta = 1$, $\eta_1 = 1$. Analysis of these curves shows that as the layer viscosity decreases, resonance frequencies shift to the right (the first curve with $\varkappa < 7$ has three resonance peaks, while the third has only one) and the resonance peaks become blurred.

Some interesting aspects are disclosed by the analysis of the problem of shear by a rigid punch of an elastic layer on an elastic Winkler foundation. Function $K(u)$ of form (1.10) satisfies here all requirements defined by (2.1), (2.2), and (3.2). It is, thus, possible to obtain an idea of wave properties of stresses under the punch and of displacement away from it by constructing the phase plane, as was done in /6/. Setting for simplicity $\rho_* = \rho h$ as above, we vary parameters $\varkappa$ and θ within wide limits. Curves showing the dependence of stress wave phase velocities under the punch and of displacement wave phase velocities on $\varkappa$ are shown in Fig.3 by dash and solid lines, respectively, for $\theta = 25$. It will be seen that when $\theta \gg 1$ and $\theta < \varkappa^2$, the zeros and poles have the same properties as in the problem with a rigid base. But the first two curves (poles and zeros) sharply increase in comparison with other curves and rapidly converge. This and the structure of formulas (2.6) and (2.7) imply that waves of small amplitude but high phase velocity appear both under and away from the punch. This is a characteristic of the type of foundation considered here.

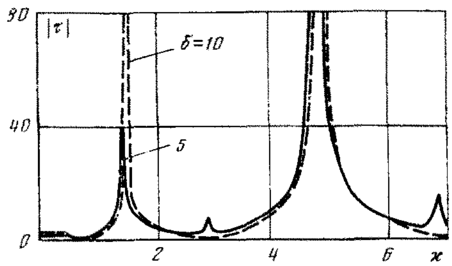

Fig.1

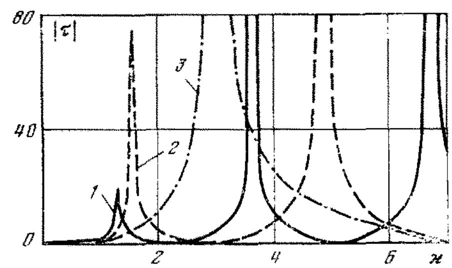

Fig.2

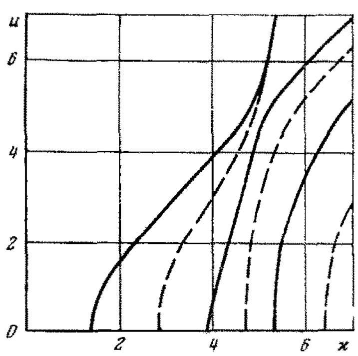

Fig.3

Similar numerical results can also be obtained for problems of a rigid punch vibration on a viscoelastic (elastic) layer on a viscoelastic (elastic) Winkler foundation. The amplitude of normal stresses under the punch and of normal waves away from it can be calculated by formulas (2.7) and (3.4), and (4.3) and (4.6), respectively.

We point out in concluding that the methods expounded in /1,2,6,8,10/ make it possible to treat in the same way problems of twist vibration and impression vibration induce by a punch pressed into a viscoelastic layer lying on a viscoelastic Winkler foundation.

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